Estimation and analysis of the reachability region in bilinear mechanical systems*

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Abstract. In this paper, the Pontryagin's maximum principle is used to identify the reachability region when a bilinear system with controllable damping and under the influence of a constant force is taken to the origin. The bilinear control problem is partitioned into simpler linear problems. Then, the optimal trajectory is obtained from the adequate combination of the partial solutions. It is also shown that the minimum time is obtained by reaching the origin with the damping active.

1 Introduction

Optimal control is an important area of control theory, which allows designing more efficient and sophisticated systems. Optimal control studies the problem of finding a control law for a given system such that a certain cost function is minimized [1].

Optimization techniques have been widely studied by many authors as Pontryagin [2], Bellmann [3], Bryson and Ho [4], Afanas'ev [5], among many other mathematicians and scientists.

The maximum principle was given by L. S. Pontryagin in 1956. But the mathematical proof appeared a few years later in several works of V. G. Boltyanskii and R. V. Gamkrelidze [2]. This methodology gives necessary conditions to find the optimal control and optimal trajectories considering the initial states of the system.

In previous paper [1] and [5] of one of the authors, the optimal control in bilinear system with controllable damping has been studied. The main result of this paper is that the optimal control of damping has two commutations at most, but excitation force has only one commutation, if any.

It is important to remark that the models considered in these works can be used to analyze fuel optimization in aircraft or automotive systems. The reader

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is referred to [6] and [7], where extensive lists of applications for bilinear systems are presented.

Besides, many real-life processes can be analyzed through linear approximations. For instance, aerospace and marine navigation systems, thermic systems, biology systems, chemistry and physics systems, robotics, among many others; can be reasonably approximated by linear systems, at least locally [8, 9]. For that reason, we can expect a good performance of the linear controller at least in the approximation region.

In [10], the authors derive a practical approach to obtain optimal controllers based on the maximum principle. This approach is based on the integration of optimal trajectory in inverse time with initial conditions in the origin at arriving time T. In that work, it was considered a system with two controllers, showing that one of the controllers has two commutations at most, while the other one presents one commutation, if any.

The present paper considers a bilinear system subject to constant external force with controllable damping. Therefore, only one control is available. In that sense, the reachability region is estimated and analyzed.

The rest of the paper is organized as follows. Section 2 presents the problem of constructing the optimal controller for systems with controllable damping under the influence of constant force. In sections 3 and 6 the problems without damping and constant damping are analyzed, respectively. The estimation and analysis of reachability region are given in section 5. Finally, some conclusions are drawn in section 6

2 System with controllable damping subject to constant force

Consider the dimensionless bilinear system of one degree of freedom of a material point

$$\ddot{x} + cu\dot{x} = 1, \tag{1}$$

with $c > 0$ as the controllable damping, u as the control, and initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$.

The control goal is to find the reachability region and valid input $u(t)$ such that

$$0 \leq u(t) \leq 1,$$

and the system (1) converges to the origin, i.e., $x(T) = 0$, $\dot{x}(T) = 0$; where T is the minimum possible time.

This bilinear system can be rewritten as follows

$$\dot{x}_1 = x_2 \tag{2}$$

$$\dot{x}_2 = 1 - cux_2, \tag{3}$$

where $x_1 = x$, $\dot{x}_1 = x_2 = \dot{x}$ and $\dot{x}_2 = \ddot{x}$.

The Hamiltonian can be defined as

$$H(x, u, \psi) = \psi_0 + \psi_1 x_2 + \psi_2(1 - cux_2),$$

while the adjoint variable vector $\psi = [\psi_1 \;\; \psi_2]^T$ is obtained from

$$\dot{\psi}_i = -\frac{\partial H}{\partial x_i}. \tag{4}$$

On the other hand, the optimal control is obtained as follows [5, 9]

$$u = \max_{0 \le u \le 1} (H) = \max_{0 \le u \le 1} (-c\psi_2 u x_2) = \frac{1 - \text{sign}(\psi_2 x_2)}{2}. \tag{5}$$

It is clear that control u has a commutation if adjoint variable ψ_2 or velocity x_2 pass through zero.

Consequently, the optimal trajectory is determined when the following aggregate nonlinear system is solved

$$\dot{x}_1 = x_2 \tag{6}$$

$$\dot{x}_2 = 1 + c\frac{1 - \text{sign}(\psi_2 x_2)}{2} x_2 \tag{7}$$

$$\dot{\psi}_1 = 0 \tag{8}$$

$$\dot{\psi}_2 = -\psi_1 + c\psi_2 \frac{1 - \text{sign}(\psi_2 x_2)}{2} \tag{9}$$

subject to $x_1(0) = x_0$, $x_2(0) = \dot{x}_0$, $x_1(T) = 0$ and $x_2(T) = 0$, where T is the arriving time to the origin.

As can be easily seen, the search for T must to be performed before the latter system can be solved.

However, because of the existence of only one control, the reachability region is reduced. This fact will be analyzed through the following sections by partitioning the original problem in linear problems, using a similar method to that presented in [10].

3 System without damping subject to constant force

Consider the dimensionless linear system of one degree of freedom without damping

$$\ddot{x} = 1, \tag{10}$$

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, which can be rewritten as

$$\dot{x}_1 = x_2 \tag{11}$$

$$\dot{x}_2 = 1, \tag{12}$$

where $x_1 = x$, $\dot{x}_1 = x_2 = \dot{x}$ and $\dot{x}_2 = \ddot{x}$.

Notice that this system is not controllable. Therefore, initial state must satisfy certain conditions in order to reach the origin. Such conditions will be clarified below.

The solution for system (11)–(12) is

$$x_1(t) = \frac{t^2}{2} + \dot{x}_0 t + x_0 \tag{13}$$

$$x_2(t) = t + \dot{x}_0. \tag{14}$$

The curves crossing through the origin are defined by integrating equations (11)–(12) in inverse time and by assuming that at the arriving time (T), the system is at the origin, i.e., $x(T) = \dot{x}(T) = 0$. Thus,

$$x_{c1}(t) = \frac{(t-T)^2}{2} \tag{15}$$

$$x_{c2}(t) = (t-T). \tag{16}$$

We use x_{c*} to describe arrival curves, which later will become in commutation curves for suitable values of $*$.

Consequently, system (10) can only arrive to the origin when initial state are on curves (15)–(16). These are the conditions mentioned earlier.

4 System with damping and constant external force

Consider now the dimensionless linear system of one degree of freedom with constat damping

$$\ddot{x} + c\dot{x} = 1, \tag{17}$$

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, and where $c > 0$ is the constant damping of the system.

This system can be rewritten as

$$\dot{x}_1 = x_2 \tag{18}$$

$$\dot{x}_2 = 1 - cx_2, \tag{19}$$

where $x_1 = x$, $\dot{x}_1 = x_2 = \dot{x}$ and $\dot{x}_2 = \ddot{x}$.

Again, system (17) is not controllable. Therefore, initial state must satisfy similar conditions to those given before in order to reach the origin.

The solution for system (18)–(19) is

$$x_1(t) = \frac{t}{c} - \frac{c\dot{x}_0 - 1}{c^2} e^{-ct} + \left(x_0 + \frac{c\dot{x}_0 - 1}{c^2} \right) \tag{20}$$

$$x_2(t) = \frac{1}{c} + \frac{c\dot{x}_0 - 1}{c} e^{-ct}, \tag{21}$$

Once more, the arriving lines are obtained by integrating equations (18)–(19) using inverse time and by assuming that at instant T (arriving time), the system is at the origin. Resulting,

$$x_{c1}(t) = \frac{(t-T)}{c} + \frac{e^{-c(t-T)} - 1}{c^2} \tag{22}$$

$$x_{c2}(t) = \frac{1}{c}(1 - e^{-c(t-T)}), \tag{23}$$

with $x_{c\bullet}$ defined as above.

As consequence, system (17) can only arrive to the origin when initial state are on curves (22)–(23).

With this in mind, the problem of finding the optimal control for system (2)–(3) is reduced to find the correct combination of the partial solutions derived so far.

5 Reachability region and optimal trajectories

In this section, the results obtained earlier are applied to determine the reachability region and the optimal trajectory in order to take system (1) to the origin.

Figure 1 shows the reachability region lines for systems without damping ($c = 0$) and with damping $c = 4$. From this graphic, it can be readily concluded that the origin can be only reach by a system with initial conditions between curves without damping and with damping. This is due to the existence of only one controller in system 1, which is the controllable damping.

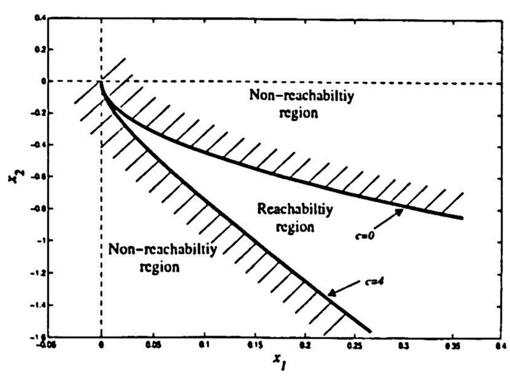

Fig. 1. Reachability and non-reachability regions for systems without damping $c = 0$ and with damping $c = 4$.

In words, if system (1) starts outside the reachability region we can only activate (or deactivate) the controllable damping. As a result, the trajectory will be parallel to the arrival curve for the system with damping (or without damping). Therefore, although velocity x_2 will be zero at some instant, the displacement x_1 will not be zero; and the system would not reach the origin.

However, if system (1) starts inside the reachability region we have two options:

1) At start, to deactivate the controllable damping. Then, wait for the intersection of the trajectory (for this case, curve without damping) and the curve corresponding to the system with damping. At this very moment, the controllable damping must to be activated. In this case the systems reach the origin on the curve with damping.
2) At start, to activate the controllable damping. Then, wait for the intersection of the trajectory (for this case, curve with damping) and the curve corresponding to a damping-less system. At that very moment, the controllable damping has to be deactivated. In this case the systems reach the origin on the curve without damping.

Then, there are two trajectories for every staring point inside the reachability region. This means that the system can arrive to the origin with $u = 0$ or $u = 1$. In order to find the commutation time t_1 and arrival time T, we use a similar approach to that presented in [10].

To determine t_1 and T, while reaching the origin on the curve with constant damping, we need to solve the simultaneous equations generated by matching equations (13), (22) and (14), (23), i.e.,

$$\frac{t_1^2}{2} + \dot{x}_0 t_1 + x_0 = \frac{(t_1 - T)}{c} + \frac{e^{-c(t_1-T)} - 1}{c^2} \tag{24}$$

$$t_1 + \dot{x}_0 = \frac{1}{c}(1 - e^{-c(t_1-T)}), \tag{25}$$

with t_1 and T as unknowns.

After simple algebraic manipulation from equation (25), we have

$$1 - e^{(-c(t_1-T))} = ct_1 + c\dot{x}_0, \tag{26}$$

then, substituting (26) in (24), yields

$$\frac{t_1^2}{2} + \dot{x}_0 t_1 + x_0 = \frac{(t_1 - T)}{c} - \frac{ct_1 + c\dot{x}_0}{c^2} \tag{27}$$

$$= \frac{(t_1 - T)}{c} - \frac{t_1}{c} - \frac{\dot{x}_0}{c}. \tag{28}$$

Finally,

$$T = -\frac{ct_1^2}{2} - c\dot{x}_0 t_1 - cx_0 - \dot{x}_0, \tag{29}$$

$$t_1 = \frac{1}{c}\left(1 - e^{-c(t_1 + \frac{ct_1^2}{2} + c\dot{x}_0 t_1 + cx_0 + \dot{x}_0)}\right). \tag{30}$$

Now, equations (29) and (30) can be solved by means of any mathematical software.

Likewise, if we want to reach the origin on curve without damping, then t_1 and T can be determined by solving simultaneous equations generated by matching equations (15), (20) and (16), (21), i.e.,

$$\frac{t_1}{c} - \frac{c\dot{x}_0 - 1}{c^2} e^{-ct_1} + \left(x_0 + \frac{c\dot{x}_0 - 1}{c^2} \right) = \frac{(t_1 - T)^2}{2} \tag{31}$$

$$\frac{1}{c} + \frac{c\dot{x}_0 - 1}{c} e^{-c_1 t} = t_1 - T. \tag{32}$$

Thus, substituting (32) in equation (31) gives

$$T = t_1 - \frac{1}{c} - \frac{c\dot{x}_0 - 1}{c} e^{-c_1 t}, \tag{33}$$

$$\frac{t_1}{c} = \frac{c\dot{x}_0 - 1}{c^2} e^{-ct_1} - \left(x_0 + \frac{c\dot{x}_0 - 1}{c^2} \right) + \frac{\left(\frac{1}{c} + \frac{c\dot{x}_0 - 1}{c} e^{-c_1 t}\right)^2}{2}. \tag{34}$$

As in the previous case, these equations can be solved using any mathematical tool.

Now, we are able to compute the trajectories for any staring point inside the reachability region.

Considering system (1) with $c = 4$ and initial conditions $x_0 = 0.2$ and $\dot{x}_0 = -0.8$. The trajectory to the origin, using the constant damping arrival curve, appears in figure 2. For this case, $t_1 = 0.1691$ and $T = 0.4839$.

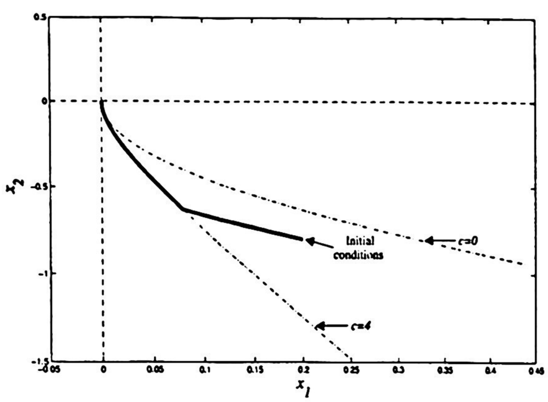

Fig. 2. Optimal trajectory with $c = 4$, $x_0 = 0.2$ and $\dot{x}_0 = -0.8$, reaching the origin on the curve for constant damping.

Now, we solve the problem under the same conditions, but assuming that the system reach the origin without damping. The trajectory for this situation is depicted in figure 3. Under this configuration, $t_1 = 0.0653$ and $T = 0.6238$. This suggests that the best option reaching the origin with damping. The following table is intended to illustrate this fact.

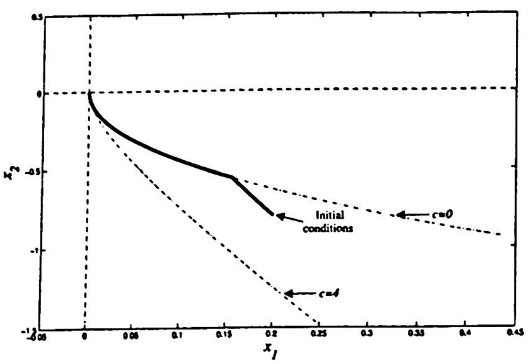

Fig. 3. Another possible trajectory with $c = 4$, $x_0 = 0.2$ and $\dot{x}_0 = -0.8$, reaching the origin without damping.

Table 1 shows the arrival times for different initial conditions inside the reachability region, when the reaching curve is either without damping or with constant damping.

Table 1. Optimal time for system (1) with $c = 4$.

	Reaching curve	
Initial conditions	without damping	constant damping
$x(0) = 0.11031; \dot{x}(0) = -0.8$	0.3590	0.3587
$x(0) = 0.15; \dot{x}(0) = -0.8$	0.5198	0.4106
$x(0) = 0.2; \dot{x}(0) = -0.8$	0.6238	0.4839
$x(0) = 0.25; \dot{x}(0) = -0.8$	0.7049	0.5719
$x(0) = 0.3; \dot{x}(0) = -0.8$	0.7744	0.6947
$x(0) = 0.32; \dot{x}(0) = -0.8$	0.8	0.7999

Notice that in every case, reaching the origin with the controllable damping activated results in the minimum time [1, 5, 10].

After a little deeper analysis, it can be deduced that if the initial conditions are on one of the arriving curves then the system cannot leave such line in order to reach the origin.

It is also important to remark that the first row of table 1 corresponds to initial conditions very close of the arriving curve with constant damping $c = 4$; while the last row corresponds to initial condition very close of the arriving curve without damping. Therefore, for the same initial velocity ($\dot{x}_0 = -0.8$ in this case), the minimum time is achieved when initial conditions are on the constant damping arrival curve, from table 1 the shortest arrival time is around $T = 0.3587$. On the other hand, the longest time is around $T = 0.8$, and it is obtained when the initial conditions are on the arrival curve without damping. So, if equation 1 represents an automobile brake system and the initial conditions are inside the reachability region. It can be concluded that applying the brakes at the end of the trajectory will result in stopping the vehicle after traveling the desired distance using the minimum time.

Thus, for a given $\dot{x}_0 < 0$, we have a reachability region $x_{\min}(0) \leq x(0) \leq x_{\max}(0)$, as shown in figure 4.

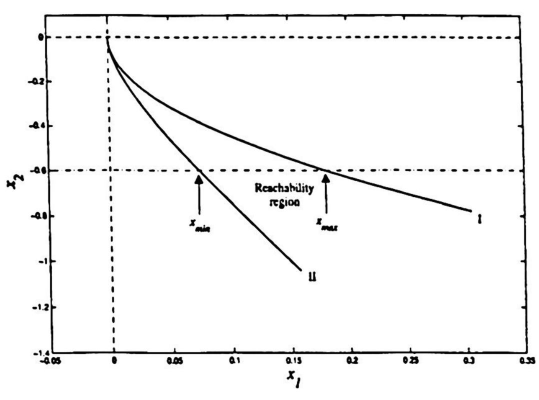

Fig. 4. Reachability region for a given velocity $\dot{x}_0 < 0$.

Therefore, on curve I (without damping) $x(0) = \frac{\dot{x}(0)}{2}$, from (15) and (16). While, on curve II (with constant damping) $x(0) = -\frac{\dot{x}(0)}{c} - \frac{1}{c^2}\ln(1 - c\dot{x}(0))$, from (22) and (23). Resulting $-\frac{\dot{x}(0)}{c} - \frac{1}{c^2}\ln(1 - c\dot{x}(0)) \leq x(0) \leq \frac{\dot{x}(0)}{2}$, for $\dot{x}(0) < 0$.

This equation allows to compute valid values of $x(0)$ in order to start inside the reachability region for a given velocity $\dot{x}(0) < 0$ and controllable damping $c > 0$. Table 2 summarizes some results for different values of $\dot{x}(0)$ and c.Now, it is clear that for a given $\dot{x}(0)$, $x_{\min}(0) \to 0$ as $c \to \infty$.

Table 2. Estimation of reachability region.

c	1		2		10	
$\dot{x}(0)$	$x_{\min}(0)$	$x_{\max}(0)$	$x_{\min}(0)$	$x_{\max}(0)$	$x_{\min}(0)$	$x_{\max}(0)$
-0.5	0.0945	0.125	0.0767	0.125	0.0320	0.125
-1	0.3069	0.5	0.2253	0.5	0.0760	0.5
-10	7.6021	50	4.2389	50	0.9538	50

6 Conclusions

In this paper, it has been analyzed the reachability region for a bilinear system subject to constant force with controllable damping. The study was carried out on the basis of the decomposition of the original problem in two parts, namely: 1) systems without damping subject to constant force and 2) systems with constant damping subject to constant force. The partition considered allows determining the reachability region in a simpler and practical way. Figures and tables illustrate the validity of the approach. It was also shown that the activation of the controllable damping at the end of the trajectory results in minimum arriving time. Another important outcome of this analysis is that the inclusion of controllable damping in control systems may increase the durability of the system, because the damping component is not active during the entire trajectory of the material point.

References

1. V. B. Kolmanovskii and V. R. Nosov, "Control in one bilinear system," *Izvestia AN USSR, Tech. Cybern.*, no. 9, 1990.
2. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidge, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes.* New York: Interscience Publishers, Inc., 1962.
3. R. E. Bellman and S. E. Dreyfus, *Applied Dynamic Programming.* Princeton, New Jersey: Princeton University press, 1962.
4. A. E. Bryson and Y. C. Ho, *Applied Optimal Control.* Toronto-London: Blaisdell Publishers, 1969.
5. V. N. Afanas'ev, V. B. Kolmanovskii, and V. R. Nosov, *Mathematical Theory of Control Systems Design.* Dordrecht/Boston/London: Kluwer Academic Publishers, 1996.
6. D. L. Elliott, *Bilinear Systems in Encyclopedia of Electrical Engineering.* New York: Wiley, 1999.
7. R. R. Mohler, *Nonlinear Systems: Volume I, Dynamics and Control, Volume II, Applications to Bilinear Control.* Englewood Cliffs, New Jersey: Prentice-Hall, 1991.
8. T. Kailath, *Linear Systems.* New Jersey: Prentice–Hall, Inc., 1980.
9. G. M. Siouris, *An Engineering Approach to Optimal Control and Estimation Theory.* New York: John Wiley & Sons, Inc., 1992.
10. V. R. Nosov, J. A. Meda-Campaña, and J. C. Gomez-Mancilla, "Optimal control of external force and damping in one mechanical system," *5th CCE 2008 (To appear).*